The Discrete Laplacian of a Rectangular Grid

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Abstract

On the results of finding the eigenvalue-eigenvector pairs of the discrete Laplacian of a rectangular m*n grid.

1 Introduction

The original motivation of this paper was to prove the conjectures of Timothy Bantham's 2006 REU paper, *The Discrete Laplacian and the Hotspot Conjecture*. Bantham's paper was motivated by the continuous hotspot conjecture of Jeff Rauch(1974). However, the conjecture was proven to be false. Bantham's paper conjectures on the second eigenvector of the Laplacian of a rectangular grid. However, there have been issues with creating a meaningful analogue of the Neumann boundary conditions for the discrete case. Thus, the original intent of proving the hotspot conjecture has put on hold for this paper, and will discuss upon the second eigenvalue-eigenvector pair of a rectangular grid. All in all, Timothy Bantham's conjecture 3.1 has been shown to be correct with a more complete proof, conjecture has been proven to be correct, and conjecture 3.3 proven false, but almost correct.

2 Definitions

Rectangular grid, *R(m,n)*: A graph connected such that it has 4 corner vertices which have two edges each, m-2 vertices that have 3 edges which make up the "short edge of a rectangle" and n-2 vertices that have 3 edges each which make up the "long edge of a rectangle" and (n-2)(m-2) inner vertices which each have four edges. When n=m, we shall call it a square, *S(n)*.

Discrete Laplacian: A matrix L such that $L_{i,j} = -1$ if $i \neq j$ and there is an edge between vertices i and j and $= 0$ if there is no edge. $L_{i,i} = -\Sigma L_{i,j}$. This is essentially the same as a Kirchoff matrix.

Second Eigenvalue: The second smallest eigenvalue.

Fiedler Vector: The eigenvector associated with the second eigenvalue.

3 Set up

The vertices of the *R(m,n)* will be enumerated as such. Imagining the rectangular grid with the longest sides running horizontally, the upper left most vertex shall be numbered 1, the vertex to vertex right as 2, and so on until we reach the rightmost vertex of that row, which shall be numbered as n. Then, the process shall be repeated with the row beneath the current row, starting again at the leftmost, and so on. The essential part of this numbering is that it is numbered by rows, so that the eigenvector can be split up into "row sections".

Using this numbering technique, our laplacian matrix K looks like

$$K = \begin{bmatrix} D_1 & -I & & & 0 \\ -I & D_2 & \ddots & & \\ & & \ddots & & \\ & & \ddots & D_2 & -I \\ 0 & & & -I & D_1 \end{bmatrix} \tag{1}$$

and

$$D_1 = \begin{bmatrix} 2 & -1 & & & 0 \\ -1 & 3 & \ddots & & \\ & & \ddots & & \\ & & \ddots & 3 & -1 \\ 0 & & & -1 & 2 \end{bmatrix} D_2 = \begin{bmatrix} 3 & -1 & & & 0 \\ -1 & 4 & \ddots & & \\ & & \ddots & & \\ & & \ddots & 4 & -1 \\ 0 & & & -1 & 3 \end{bmatrix}$$

(Note K is $n*m$ x $n*m$ and D_1, D_2 are both nxn and $m < n$ for a rectangular grid) An example matrix is shown in Bantham's paper.

Now, to solve for the eigenvalues and eigenvectors, we must solve this equation.

$$K\Phi = \lambda\Phi \tag{2}$$

Since K is a laplacian matrix, it is clear that 0 is an eigenvalue, and since the rectangular grid is connected, hence there is only one connected component, the second eigenvalue will be non-zero.

Theorem: *The eigenvalues of the laplacian matrix for R(m,n) are of the form*

$$\lambda_{k,l} = (1 - \frac{cos(\frac{3\pi k}{2n})}{cos(\frac{\pi k}{2n})}) + (1 - \frac{cos(\frac{3\pi l}{2m})}{cos(\frac{\pi l}{2m})}) \tag{3}$$

Let $\theta = \frac{\pi k}{n}$ and $\psi = \frac{\pi l}{m}$

This simplifies to

$$\lambda_{k,l} = (2sin\frac{\theta}{2})^2 + (2sin\frac{\psi}{2})^2 \tag{4}$$

with eigenvectors as

$$\Phi^{k,l} = (\phi(\vec{x,y})) = \begin{pmatrix} \begin{pmatrix} \phi(1,1) \\ \phi(2,1) \\ \vdots \\ \phi(n,1) \end{pmatrix} \\ \begin{pmatrix} \phi(1,2) \\ \phi(2,2) \\ \vdots \\ \phi(n,2) \end{pmatrix} \\ \vdots \\ \begin{pmatrix} \phi(1,m) \\ \phi(2,m) \\ \vdots \\ \phi(n,m) \end{pmatrix} \end{pmatrix} \tag{5}$$

where

$$\phi(x,y) = cos(\frac{\pi}{n}k(x-\frac{1}{2}))cos(\frac{\pi}{m}l(y-\frac{1}{2})) \tag{6}$$

3.1 Proof

Written out, (7) becomes

$$\begin{aligned}
D_1\Phi_{1\sim n,1} - I\Phi_{1\sim n,2} &= \lambda\Phi_{1\sim n,1}\\
-I\Phi_{1\sim n,1} + D_2\Phi_{1\sim n,2} - I\Phi_{1\sim n,3} &= \lambda\Phi_{1\sim n,2}\\
&\vdots\\
-I\Phi_{1\sim n,m-2} + D_2\Phi_{1\sim n,m-1} - I\Phi_{1\sim n,m} &= \lambda\Phi_{1\sim n,m-1}\\
-I\Phi_{1\sim n,m} + D_1\Phi_{1\sim n,m} &= \lambda\Phi_{1\sim n,m}
\end{aligned}$$

Where it is understood that $\Phi_{1\sim n,1}$ and similar terms are equivalent to $(\phi(1,1),\phi(1,1),\cdots,\phi(n,1))^T$ and the like, where $\phi_(x,y) = cos(\frac{\pi}{n}k(x-\frac{1}{2}))cos(\frac{\pi}{m}l(y-\frac{1}{2}))$

Let us observe $D_1\Phi_{1\sim n,1} - I\Phi_{1\sim n,2} = \lambda\Phi_{1\sim n,1}$ which when written out is

$$\begin{aligned}
2\phi(1,1) - \phi(2,1) - \phi(1,2) &= \lambda\phi(1,1)\\
-\phi(1,1) + 3\phi(2,1) - \phi(3,1) - \phi(2,2) &= \lambda\phi(2,1)\\
&\vdots\\
-\phi(n-2,1) + 3\phi(n-1,1) - \phi(n,1) - \phi(n-1,2) &= \lambda\phi(n-1,1)\\
-\phi(n-1,1) + 2\phi(n,1) - \phi(n,2) &= \lambda\phi(n,1)
\end{aligned}$$

To show that $\Phi^{k,l}$ is in fact an eigenvalue, we must show that the above equalities hold.

Let us start with the base case, i.e. the first line.

$$\begin{aligned}
2\phi(1,1) - \phi(2,1) - \phi(1,2) &= \lambda\phi(1,1)\\
2cos(\frac{\theta}{2})cos(\frac{\psi}{2}) - cos(\frac{3\theta}{2})cos(\frac{\psi}{2}) - cos(\frac{\theta}{2})cos(\frac{3\psi}{2}) &= \lambda cos(\frac{\theta}{2})cos(\frac{\psi}{2})\\
\frac{2cos(\frac{\theta}{2})cos(\frac{\psi}{2}) - cos(\frac{3\theta}{2})cos(\frac{\psi}{2}) - cos(\frac{\theta}{2})cos(\frac{3\psi}{2})}{cos(\frac{\theta}{2})cos(\frac{\psi}{2})} &= \lambda\\
2 - \frac{cos(\frac{3\theta}{2})}{cos(\frac{\theta}{2})} - \frac{cos(\frac{3\psi}{2})}{cos(\frac{\psi}{2})} &= \lambda\\
(1 - \frac{cos(\frac{3\theta}{2})}{cos(\frac{\theta}{2})}) + (1 - \frac{cos(\frac{3\psi}{2})}{cos(\frac{\psi}{2})}) &= \lambda
\end{aligned}$$

We see that this agrees with our conjectured $\lambda_{k,l}$ Now for the other boundary condition, i.e. the last line.

$$
\begin{aligned}
-\phi(n-1,1)+2\phi(n,1)-\phi(n,2) &= \lambda\phi(n,1)\\
-cos(\frac{\pi k}{n}(n-1-\frac{1}{2}))cos(\frac{\frac{\pi l}{m}}{2})+2cos(\frac{\pi k}{n}(n-\frac{1}{2}))cos(\frac{\frac{\pi l}{m}}{2})-cos(\frac{\pi k}{n}(n-\frac{1}{2}))cos(\frac{\pi l}{m}(2-\frac{1}{2})) &= \lambda\phi(n,1)\\
-cos(\frac{\pi k}{n}(n-\frac{3}{2}))cos(\frac{\frac{\pi l}{m}}{2})+2cos(\frac{\pi k}{n}(n-\frac{1}{2}))cos(\frac{\frac{\pi l}{m}}{2})-cos(\frac{\pi k}{n}(n-\frac{1}{2}))cos(\frac{\pi l}{m}(\frac{3}{2})) &= \lambda\phi(n,1)\\
-cos(\pi k-\frac{3\pi k}{2n})cos(\frac{\pi l}{2m})+2cos(\pi k-\frac{\pi k}{2n})cos(\frac{\pi l}{2m})-cos(\pi k-\frac{\pi k}{2n})cos(\frac{3\pi l}{2m}) &= \lambda\phi(n,1)\\
-(-1)^k cos(\frac{3\pi k}{2n})cos(\frac{\pi l}{2m})+2(-1)^k cos(\frac{\pi k}{2n})cos(\frac{\pi l}{2m})-(-1)^k cos(\frac{\pi k}{2n})cos(\frac{3\pi l}{2m}) &= \lambda\phi(n,1)
\end{aligned}
$$

Since $\phi(n,1)=(-1)^k cos(\frac{\pi k}{2n})cos(\frac{\pi l}{2m})$, we have

$$
\begin{aligned}
\lambda &= \frac{-(-1)^k cos(\frac{3\pi k}{2n})cos(\frac{\pi l}{2m})+2(-1)^k cos(\frac{\pi k}{2n})cos(\frac{\pi l}{2m})-(-1)^k cos(\frac{\pi k}{2n})cos(\frac{3\pi l}{2m})}{(-1)^k cos(\frac{\pi k}{2n})cos(\frac{\pi l}{2m})}\\
&= 2-\frac{cos(\frac{3\pi k}{2n})}{cos(\frac{\pi k}{2n})}-\frac{cos(\frac{3\pi l}{2m})}{cos(\frac{\pi l}{2m})}
\end{aligned}
$$

This agrees with $\lambda_{k,l}$ as well

Now, on to the general case for $2 \leq r \leq n-1$

$$
\begin{aligned}
-\phi(r-2,1)+3\phi(r-1,1)-\phi(r,1)-\phi(r-1,2) &= \lambda\phi(r-1,1)\\
-cos(\theta(r-\frac{5}{2}))cos(\frac{\psi}{2})+3cos(\theta(r-\frac{3}{2}))cos(\frac{\psi}{2})-cos(\theta(r-\frac{1}{2}))cos(\frac{\psi}{2})-cos(\theta(r-\frac{3}{2}))cos(\frac{3\psi}{2}) &= \lambda\phi(r-1,1)\\
\frac{-cos(\theta(r-\frac{5}{2}))cos(\frac{\psi}{2})+3cos(\theta(r-\frac{3}{2}))cos(\frac{\psi}{2})-cos(\theta(r-\frac{1}{2}))cos(\frac{\psi}{2})-cos(\theta(r-\frac{3}{2}))cos(\frac{3\psi}{2})}{cos(\theta(r-\frac{3}{2}))cos(\frac{\psi}{2})} &= \lambda\\
\frac{-cos(\theta(r-\frac{5}{2}))}{cos(\theta(r-\frac{3}{2}))}+3-\frac{cos(\theta(r-\frac{1}{2}))}{cos(\theta(r-\frac{3}{2}))}-\frac{cos(\frac{3\psi}{2})}{cos(\frac{\psi}{2})} &= \lambda
\end{aligned}
$$

Using the Cosine sum-to-product formula, we have
$cos(\theta(r-\frac{5}{2}))+cos(\theta(r-\frac{1}{2}))=2cos(\theta(r-\frac{3}{2}))cos(\theta)$
Furthermore
$2cos(\theta)-1=\frac{cos(\frac{3\theta}{2})}{cos(\frac{\theta}{2})}$

$$
\begin{aligned}
\frac{-cos(\theta(r-\frac{5}{2}))}{cos(\theta(r-\frac{3}{2}))}+3-\frac{cos(\theta(r-\frac{1}{2}))}{cos(\theta(r-\frac{3}{2}))}-\frac{cos(\frac{3\psi}{2})}{cos(\frac{\psi}{2})} &= \lambda\\
3-(\frac{cos(\frac{3\theta}{2})}{cos(\frac{\theta}{2})}+1)-\frac{cos(\frac{3\psi}{2})}{cos(\frac{\psi}{2})} &= \lambda\\
2-\frac{cos(\frac{3\theta}{2})}{cos(\frac{\theta}{2})}-\frac{cos(\frac{3\psi}{2})}{cos(\frac{\psi}{2})} &= \lambda
\end{aligned}
$$

This is consistent as well, so the case has been proven for $D_1\Phi - I\Phi = \lambda\Phi$

Now, we must prove for

$$
\begin{aligned}
-I\Phi_{1\sim n,1} + D_2\Phi_{1\sim n,2} - I\Phi_{1\sim n,1} &= \lambda\Phi_{1\sim n,2} \\
&\vdots \\
-I\Phi_{1\sim n,m-2} + D_2\Phi_{1\sim n,m-1} - I\Phi_{1\sim n,m} &= \lambda\Phi_{1\sim n,m-1} \\
-I\Phi_{1\sim n,m-1} + D_1\Phi_{1\sim n,m} &= \lambda\Phi_{1\sim n,m}
\end{aligned}
$$

The last line is already proved by using the fact that $\Phi_{1\sim n,m} = (-1)\Phi_{1\sim n,1}$ and $\Phi_{1\sim n,m-1} = (-1)\Phi_{1\sim n,2}$. Now to prove for $2 \le q \le m-1$

$$
-I\Phi_{1\sim n,q-1} + D_2\Phi_{1\sim n,q} - I\Phi_{1\sim n,q+1} = \lambda\Phi_{1\sim n,q} \tag{7}
$$

Now, note that $\Phi_{1\sim n,q} = (cos\frac{\theta}{2}, cos\frac{3\theta}{2}, \cdots, cos\frac{(2n-1)\theta}{2})^T cos\frac{(2q-1)\psi}{2}$

Let $\vec{v} = (cos\frac{\theta}{2}, cos\frac{3\theta}{2}, \cdots, cos\frac{(2n-1)\theta}{2})^T$

$$
\begin{aligned}
-I\Phi_{1\sim n,q-1} + D_2\Phi_{1\sim n,q} - I\Phi_{1\sim n,q+1} &= \lambda\Phi_{1\sim n,q} \\
-cos\frac{(2q-3)\psi}{2}\vec{v} + D_2 cos\frac{(2q-1)\psi}{2}\vec{v} - cos\frac{(2q+1)\psi}{2}\vec{v} &= \lambda cos\frac{(2q-1)\psi}{2}\vec{v}
\end{aligned}
$$

Using the sum to product formula, we have
$cos\frac{(2q-3)\psi}{2} + cos\frac{(2q+1)\psi}{2} = 2cos\frac{(2q-1)\psi}{2}cos\psi$

$$
\begin{aligned}
-cos\frac{(2q-3)\psi}{2}\vec{v} + D_2 cos\frac{(2q-1)\psi}{2}\vec{v} - cos\frac{(2q+1)\psi}{2}\vec{v} &= \lambda cos\frac{(2q-1)\psi}{2}\vec{v} \\
-2cos\psi\vec{v} + D_2\vec{v} &= \lambda\vec{v} \\
-(\frac{cos(\frac{3\psi}{2})}{cos(\frac{\psi}{2})} + 1)\vec{v} + D_2\vec{v} &= \lambda\vec{v} \\
(D_2 - I)\vec{v} &= (\lambda + \frac{cos(\frac{3\psi}{2})}{cos(\frac{\psi}{2})})\vec{v} \\
D_1\vec{v} &= (\lambda + \frac{cos(\frac{3\psi}{2})}{cos(\frac{\psi}{2})})\vec{v} \\
D_1\vec{v} - I\vec{v} &= (\lambda + \frac{cos(\frac{3\psi}{2})}{cos(\frac{\psi}{2})} - 1)\vec{v}
\end{aligned}
$$

$D_1\Phi - I\Phi = \lambda\Phi$ with $\lambda_{k,l} = (1 - \frac{cos(\frac{3\theta}{2})}{cos(\frac{\theta}{2})})(1 - \frac{cos(\frac{3\psi}{2})}{cos(\frac{\psi}{2})})$ was proven above already.

Setting l=0, We have $\lambda_{k,0} = (1 - \frac{cos(\frac{3\theta}{2})}{cos(\frac{\theta}{2})})$ and $\Phi^{k0} = \vec{v} * cos(0) = \vec{v}$. Thus, we see that our above set of equations boiled down to

$\lambda = (1 - \frac{cos(\frac{3\theta}{2})}{cos(\frac{\theta}{2})}) + (1 - \frac{cos(\frac{3\psi}{2})}{cos(\frac{\psi}{2})})$.

λ agrees with $\lambda_{k,l}$, thus we know $\lambda_{k,l}$ is a general form of the eigenvalue with $\Phi^{k,l}$ as the eigenvectors. It is only necessary to show that these are the $n * m$ eigenvalue-eigenvector pairs. Notice that $\vec{v}$ can take any k value from $1, \cdots, n$, and denote these vectors as $\vec{v_i}$ for $0 \le i \le n-1$. By construction of $\vec{v_i}$ it is clear that they are independent. Now, l can take m distinct values and likewise, it will yield different values for $cos(\frac{\pi l}{m}(y - 1/2))$. Since

$\Phi^{k,l} = (cos(\frac{\pi l}{m}\frac{1}{2})\vec{v_i}, cos(\frac{\pi l}{m}\frac{3}{2})\vec{v_i}, \cdots, cos(\frac{\pi l}{m}\frac{m-3}{2}))\vec{v_i}, cos(\frac{\pi l}{m}\frac{m-1}{2})\vec{v_i})^T$, we see that there are $m * n$ independent vectors, $\Phi^{k,l}$, for $1 \le k \le n; 1 \le l \le m$.

It is also clear that there at most $m * n$ distinct values for $\lambda_{k,l}$, and since each eigenvalue is associated with an eigenvector, we see that there are $m * n$ distinct eigenvalue-eigenvector pairs. Therefore, $\Phi^{k,l}$, $\lambda_{k,l}$ are indeed the general form of the eigenvectors and eigenvalues respectively. □

Theorem: *The second eigenvalue and the an associated fiedler eigenvector for R(n,m) where $m < n$, is associated with (k,l) = (1,0)*

Proof: $\lambda_{k,l} = (2sin\frac{\pi(k)}{2n})^2 + (2sin(\frac{\pi(l)}{2m}))^2 \geq 0$ with equality if and only if $(k,l) = (0,0)$. So the smallest eigenvalue aside from $\lambda_{0,0}$ must be of the form $\lambda_{0,l}$ or $\lambda_{k,0}$.

$\lambda_{0,l} = (2sin(\frac{\pi(l)}{2m}))^2$

$\frac{d\lambda_{0,l}}{dl} = \frac{2\pi}{2m}sin(\frac{\pi l}{2m})$

This is always positive on (1,m), so the minimum value of $\lambda_{0,l}$ is attained on $\lambda_{0,1}$

$\lambda_{k,0} = (2sin\frac{\pi(k)}{2n})^2$

$\frac{d\lambda_{k,0}}{dk} = \frac{2\pi}{2n}sin(\frac{\pi k}{2n})$

Similarly is always positive on (1,n), so the minimum value of $\lambda_{k,0}$ is attained on $\lambda_{1,0}$

Comparing $\lambda_{1,0} = (2sin\frac{\pi}{2n})^2$ and $\lambda_{0,1} = (2sin(\frac{\pi}{2m}))^2$. Since $m < n$ by construction, it follows that $\lambda_{0,1} > \lambda_{1,0}$. □

4 Remark

When the eigenvalues have multiplicty greater than one, we observe different eigenvectors than those that we get from Matlab. This is because if we have two eigenvectors Φ_1, Φ_2, both with eigenvalue λ, then any linear combination of Φ_1 and Φ_2 are also solutions.

$$K(c_1\Phi_1 + c_2\Phi_2) = Kc_1\Phi_1 + Kc_2\Phi_2 = c_1\lambda\Phi_1 + c_2\lambda\Phi_2 = \lambda(c_1\Phi_1 + c_2\lambda\Phi_2)$$

5 Side Remark

For any connected graph whose laplacian L has the the laplacian of *R(2,3)* as a proper submatrix, then, by using the Cauchy-interlacing theorem, we can show that the second eigenvalue of L is at most 1, and since it is connected, we know the first eigenvalue is 0, and the second eigenvalue is greater than 0. Using this, a method to find the eigenvector and eigenvalue of L is to use the Inverse Power iteration method by letting our "approximation" of the eigenvalue be $\frac{1}{2}$.

6 Further Work

1. Explore the meanings of the other eigenvalues
2. Look at the case of m=n
3. Continue Timothy Bantham's work
4. Explore the meaning of letting m be constant and taking the limit of the second eigenvalue as n goes to infinity.

7 References

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[3] F.R.K Chung. "Spectral Graph Theory" CBMS-volume 92. American Mathematical Society, 1997
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